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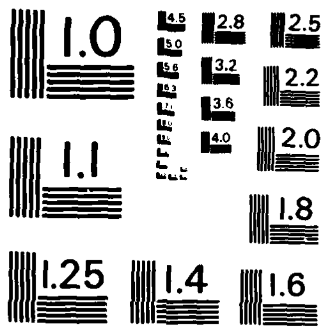

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EXTREMA AND LEVEL CROSSINGS OF χ^2 PROCESSES

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EXTREMA AND LEVEL CROSSINGS OF χ^2 PROCESSES

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ABSTRACT

This document studies
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Key words and phrases:

χ^2 processes, upcrossings, extrema, Slepian model process, Poisson limit, distribution of maximum.

1. INTRODUCTION

In this paper we shall continue our study of various sample path properties of χ^2 processes, begun in Aronowich and Adler (1985). The χ^2 process is defined, in law, via a vector $\underset{\sim}{X}(t) = (X_1(t),\ldots,X_n(t))$ of n independent, stationary, zero mean Gaussian processes as

$$(1.1) \qquad Y(t) := \underset{\sim}{X}'(t)\underset{\sim}{X}(t) = \sum_{i=1}^{n} X_i^2(t) .$$

We have already noted in the previous paper why this process is interesting, and we refer the reader to the introduction there for details. For the sake of completeness, however, let us recall here that χ^2 processes are of applied interest in a number of stochastic modelling situations, and of considerable theoretical interest as perhaps the only non-Gaussian process for which it is possible to develop a theory almost as general as in the Gaussian case. Thus the χ^2 process presents a natural candidate for studying the robustness of the Gaussian theory when similar, but non-Gaussian, processes are considered. As was the case in the previous paper, our principle concern will be to see when non-Gaussian behaviour is exhibited by χ^2 processes, rather than trying to see what has to be done to obtain almost-Gaussian phenomena. In this view we shall pay particular emphasis to low sections of the χ^2 processes (which is clearly bounded below by zero) which exhibit highly non-Gaussian behaviour, rather than emphasise the high sections, which are much closer to Gaussian. This emphasis is also related to and motivated by the applications mentioned in the earlier paper.

The paper is organised as follows. Section 2 studies the fine behaviour of χ^2 processes after level crossings and extrema. Our approach is to develop the Slepian model process in each case, this representing the conditional distribution of the original process after a level crossing or extremum. The model processes take on particularly simple forms if the level crossing or extremum is either at a high level or close to zero, and so we study these cases in detail. In all cases it turns out that at high and low extrema χ^2 processes look like random parabolas, as do their Gaussian counterparts and as, in fact, does any smooth enough function. The differences between the χ^2 and Gaussian cases show up in the parameters of these parabolas, which are generally random variables.

From the simple structure of these limiting parabolas it is a simple matter to deduce the distribution of variables such as the duration of high level excursions. Examples of this type of calculation are given in Section 3.

In Section 4 we consider the asymptotic Poisson form of the (normalized) process of downcrossings of a low level by a χ^2 process. Upcrossings have already been considered by Sharpe (1978) and Lindgren (1980a,b), and these follow the Gaussian model quite closely. Downcrossings of low levels exhibit somewhat different behaviour, primarily because of the fact that the process is bounded from below. As is well known, results of the types of Sections 2 and 4 are often combined in practice to model processes as a sequence of random parabolas positioned over the points of a Poisson process. Consequently, the results of Section 4 are of reasonable applied interest.

Given the asymptotic Poisson result it is a comparatively simple exercise to obtain the (normalized) distribution of $\inf\{Y(t): t \in [0,T]\}$ as $T \to \infty$. This is done in the final section 5.

We conclude this section with some definitions and notation.

Throughout the paper we shall assume that the component Gaussian processes in (1.1) are independent and have a common distribution, with covariance function

$$R(t) = R_X(t) := E\{X(s)X(s+t)\}.$$

We also assume, without any loss of generality, that $R(0) = 1$; i.e. the X_i all have unit variance. We shall assume that each X_i is twice continuously differentiable with probability one. (c.f. Cramer and Leadbetter (1967) and Dudley (1973) for sufficient conditions on R for this to be true.) Thus we can define the second and fourth spectral moments of the X_i, respectively, as

$$(1.2) \qquad \lambda = \lambda_X := \left.\frac{-d^2R(t)}{dt^2}\right|_{t=0} = E\{\dot{X}_i^2(t)\}$$

$$\nu = \nu_X := \left.\frac{d^4R(t)}{dt^4}\right|_{t=0} = E\{\ddot{X}_i^2(t)\},$$

where $\dot{X}$ and $\ddot{X}$ are the first two sample path derivatives of X. Furthermore, we then have the following behaviour for R and its derivatives near the origin:

$$(1.3) \qquad R(t) = 1 - \frac{\lambda t^2}{2} + \frac{\nu t^4}{4!} + o(t^4) \qquad \text{as } t \to 0,$$

$$(1.4) \qquad |\ddot{R}(t) + \lambda| \leq O(t^2) \qquad \text{as } t \to 0 .$$

It is immediate from the definition (1.1) of the χ^2 process Y that it inherits from the X_i both stationarity and twice differentiable sample paths, so that we can express its first and second order derivatives via those of the X_i. We have

$$(1.3) \qquad \dot{Y}(t) := \frac{d}{dt} Y(t) = 2 \sum_{i=1}^{n} X_i(t)\dot{X}_i(t) = 2\underset{\sim}{X}'(t)\, \dot{\underset{\sim}{X}}(t) ,$$

where $\dot{\underset{\sim}{X}}(t) = (\dot{X}_1(t),\ldots,\dot{X}_n(t))$. Similarly, writing $\ddot{\underset{\sim}{X}}(t)$ for the vector of second order derivatives of the X_i, we have

$$(1.4) \qquad \ddot{Y}(t) := \frac{d^2}{dt^2} Y(t) = 2 \sum_{i=1}^{n} [\dot{X}_i^2(t) + X_i(t)\ddot{X}_i(t)]$$

$$= 2\{\dot{\underset{\sim}{X}}'(t)\, \dot{\underset{\sim}{X}}(t) + \underset{\sim}{X}'(t)\, \ddot{\underset{\sim}{X}}(t)\}.$$

Furthermore, Y has mean n and covariance function

$$(1.5) \qquad R_Y(t) := E\{Y(s)Y(s+t)\} = 2nR_X^2(t)$$

To avoid uninteresting technicalities, we shall assume throughout the paper that all the joint (finite-dimensional) distributions of $Y, \dot{Y}$ and $\ddot{Y}$ are non-singular.

As we have already noted, considerable emphasis will be placed on studying the sample path properties of χ^2 processes via <u>model processes</u> created by conditioning the original process, in a horizontal window sense, on some event having occurred. We shall assume throughout

that the reader is familiar with this idea, and with the basic motivation and methodology behind it. A full survey of the model process approach is given in Lindgren (1984), and a background sufficient for our purposes can be found in Section 10.3 of Leadbetter, Lindgren and Rootzen (1983).

In order for the model process approach to be meaningful it is necessary to assume that Y is <u>ergodic</u>. This clearly follows from the ergodicity of the X_i processes, which, in turn, follows from

$$(1.6) \qquad R(t) \to 0 \quad \text{as } |t| \to \infty ,$$

a condition which we shall henceforth assume.

Finally, in order to compare results in the χ^2 case to corresponding results for Gaussian processes, we shall require a Gaussian process "matched", in terms of its second order properties, to our χ^2 process. Such a process, $Z(t)$ say, will be defined to have mean n, variance $2n$, and covariance function (1.5), identical to that of the χ^2 process.

2. BEHAVIOUR OF χ^2 PROCESSES AFTER A LEVEL CROSSING

In this section we study the χ^2 process $Y(t)$, $t > 0$, given that at $t=0$ the process crossed, either from above, or from below, a given level. It is now well known that the appropriate framework for such a study is via the notion of "horizontal window" (h.w.) conditioning introduced by Kac and Slepian (1959), and the so-called Slepian model process. For details of the latter we refer the reader to Lindgren's (1984) recent review.

We commence by considering $Y(t)$, conditioned, in the above sense, on having an upcrossing of the level $u > 0$ at time $t = 0$. We write $Y_u(t)$ for the conditional, or "model", process. Then, writing $\overset{L}{=}$ for equivalence in law, we have

Theorem 2.1

(i) *Following an upcrossing,*

$$(2.1)\qquad Y_u(t) \overset{L}{=} \sum_{i=1}^{n} \{\sqrt{u}R(t)V_i - \dot{R}(t)W_i/\sqrt{\lambda} + \Delta_i(t)\}^2 ,$$

where the random vectors $\underset{\sim}{V} = (V_1,\ldots,V_n)'$, $\underset{\sim}{W} = (W_1,\ldots,W_n)'$ *have the joint density*

$$(2.2)\qquad p(\underset{\sim}{v},\underset{\sim}{w}) = \pi^{\frac{1}{2}-n}2^{-(n+1)/2}\Gamma(n/2)(\underset{\sim}{v}'\underset{\sim}{w})^{+}\exp\{-\underset{\sim}{w}'\underset{\sim}{w}/2\},$$

with respect to (surface measure) $\times$ (Lebesgue measure) *on the cylinder* $(\underset{\sim}{v}'\underset{\sim}{v} = 1) \times \mathbb{R}^n$. Here $(\cdot)^+ = \max\{\cdot,0\}$, and $\Delta_i(t)$ are independent, zero mean, Gaussian processes, independent of V,W, and with the common covariance function:

$$(2.3)\qquad r(s,t) = R(s-t) - R(s)R(t) - \dot{R}(s)\dot{R}(t)/\lambda .$$

(ii) <u>The characteristic function of the joint distribution of</u> $Y_u(t_1),\ldots,Y_u(t_k)$ <u>is given by</u>

$$(2.4)\qquad \psi(\nu_1,\ldots,\nu_k) = \frac{e^{u/2}B^{-n/2}}{2\sqrt{2\pi\lambda u}} \int_0^\infty \int_{-\infty}^\infty |z| \exp\{-\frac{uA}{2B} - i\xi z\} d\xi dz$$

<u>where</u>

$$A = \det \begin{bmatrix} 1-2i\nu_1 & -2i\nu_2 R_{12} & \cdots & -2i\nu_k R_{1k} & 2i\xi\dot{R}_1 & -2i\xi R_1 \\ -2i\nu_1 R_{21} & 1-2i\nu_2 & \cdots & -2i\nu_k R_{2k} & 2i\xi\dot{R}_2 & -2i\xi R_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -2i\nu_1 R_{k1} & -2i\nu_2 R_{k2} & \cdots & 1-2i\nu_k & 2i\xi\dot{R}_k & -2i\xi R_k \\ -2i\nu_1 R_1 & -2i\nu_2 R_2 & \cdots & -2i\nu_k R_k & 1 & -2i\xi \\ 2i\nu_1\dot{R}_1 & 2i\nu_2\dot{R}_2 & \cdots & 2i\nu_k\dot{R}_k & -2i\xi\lambda & 1 \end{bmatrix}$$

$$B = \det \begin{bmatrix} 1-2i\nu_1 & -2i\nu_2 R_{12} & \cdots & -2i\nu_k R_{1k} & R_1 \\ -2i\nu_1 R_{21} & 1-2i\nu_2 & \cdots & -2i\nu_k R_{2k} & R_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -2i\nu_1 R_{k1} & -2i\nu_2 R_{k2} & \cdots & 1-2i\nu_k & R_k \\ -2i\nu_1 R_1 & -2i\nu_2 R_2 & \cdots & -2i\nu_k R_k & 1 \end{bmatrix}$$

<u>and</u> $\qquad R_j = R(t_j),\ R_{j\ell} = R(t_j - t_\ell), \dot{R}_j = \dot{R}(t_j), \dot{R}_{j\ell} = \dot{R}(t_j - t_\ell)$.

<u>Proof.</u> Part (i) of the Theorem is given in Lindgren (1980c, 1984). The distribution of $\underset{\sim}{V}$ and $\underset{\sim}{W}$ is the h.w. conditional distribution of $\underset{\sim}{X}(0)/\sqrt{u}$ and $\dot{\underset{\sim}{X}}(0)/\sqrt{\lambda}$, given an upcrossing of a level u at $t=0$.

Note that although $\underset{\sim}{X}(0)$ and $\dot{\underset{\sim}{X}}(0)$ are independent, $\underset{\sim}{V}$ and $\underset{\sim}{W}$ are not.

(ii) Under the conditions given in the Introduction, we have:

$$(2.5)\qquad p(\underset{\sim}{y}\,|\,Y(0) = 0,\ \dot{Y}(0) \geq 0)_{h.w.} = \mu^{-1}\int_0^\infty |z|\,p(\underset{\sim}{y},u,z)\,dz,$$

where $p(\underset{\sim}{y},u,z)$ is the joint density of $\underset{\sim}{Y} = (Y(t_1),\ldots,Y(t_k))'$, $Y(0)$ and $\dot{Y}(0)$, and μ is the expected number of upcrossings of the level u by $Y(t)$ in unit time. Thus

$$(2.6)\qquad \mu = \frac{\sqrt{\lambda/\pi}}{\Gamma(n/2)}\,(u/2)^{(n-1)/2}\,e^{-u/2}\ ,$$

(see Sharpe (1978)).

The joint characteristic function of $\underset{\sim}{Y}$, $Y(0)$ and $\dot{Y}(0)$ is given by:

$$\begin{aligned}
\psi(\underset{\sim}{\nu},\nu_0,\xi) &= E(\exp\{i[\underset{\sim}{\nu}'\underset{\sim}{Y} + \nu_0 Y(0) + \xi\dot{Y}(0)]\}) \\
&= E\,\exp\{i[\sum_{j=1}^{k}\nu_j\underset{\sim}{X}'(t_j)\underset{\sim}{X}(t_j) + \nu_0\underset{\sim}{X}'(0)\underset{\sim}{X}(0) + 2\xi\underset{\sim}{X}'(0)\dot{\underset{\sim}{X}}(0)]\} \\
&= \{E(\exp\, i\,[\sum_{j=1}^{k}\nu_j X_1^2(t_j) + \nu_0 X_1^2(0) + 2\xi X_1(0)\dot{X}_1(0)])\}^n \\
&= \{E(\exp\, i[(\underset{\sim}{X}_1',\ X_1(0),\dot{X}_1(0))\underset{\sim}{N}(\underset{\sim}{X}_1',X_1(0),\dot{X}_1(0))']\}^n\ ,
\end{aligned}$$

where $\underset{\sim}{X}_1 := (X_1(t_1),\ldots,X_1(t_k))'$ and

$$\underset{\sim}{N} := \begin{bmatrix} \nu_1 & & & & \\ & \ddots & & 0 & \\ & & \nu_k & & \\ 0 & & & \nu_0 & \xi \\ & & & \xi & 0 \end{bmatrix}.$$

The covariance matrix of the normal variates $\underset{\sim}{X}_1, X_1(0), \dot{X}_1(0)$ is (c.f. (1.2))

$$(2.7)\qquad \underset{\sim}{M} = \begin{bmatrix} 1 & R_{12} & \cdots & R_{1k} & R_1 & -\dot{R}_1 \\ R_{21} & 1 & \cdots & R_{2k} & R_2 & -\dot{R}_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{k1} & R_{k2} & \cdots & 1 & R_k & -\dot{R}_k \\ R_1 & R_2 & \cdots & R_k & 1 & 0 \\ -\dot{R}_1 & -\dot{R}_2 & \cdots & -\dot{R}_k & 0 & \lambda \end{bmatrix} .$$

Therefore (see, for example, Lukacs and Laha (1964)),

$$\psi(\underset{\sim}{\nu}, \nu_0, \xi) = \{\det(I-2i\, \underset{\sim}{M}\underset{\sim}{N})\}^{-n/2} ,$$

and a simple calculation shows that

$$\det(I-2i\, \underset{\sim}{M}\underset{\sim}{N}) = A - 2i\nu_0 B ,$$

where A and B are given after (2.4).

Using the inversion theorem for Fourier transforms, we now have

$$(2.8)\qquad p(\underset{\sim}{y},u,z) = (2\pi)^{-k-2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\psi(\underset{\sim}{\nu},\nu_0,\xi)\exp\{-i[\underset{\sim}{\nu}'\underset{\sim}{y}+\nu_0 u+\xi z]\}d\xi \prod_{j=0}^{k} d\nu_j .$$

$$= (2\pi)^{-k-2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i\underset{\sim}{\nu}'\underset{\sim}{y}}\int_{-\infty}^{\infty} e^{-i\xi z}\int_{-\infty}^{\infty}(A-2i\nu_0 B)^{-n/2} e^{-iu\nu_0} d\nu_0 d\xi \prod_{j=1}^{k} d\nu_j$$

To evaluate the innermost integral we note that

$$(2.9) \qquad \int_{-\infty}^{\infty}(\beta+ai\nu_0)^{-n/2}e^{-iu\nu_0}\,d\nu_0 = \begin{cases} 0 & , \; a=+1 \\ 2\pi u^{n/2-1}e^{-u\beta}/\Gamma(n/2) & , \; a=-1 \end{cases}$$

for $\operatorname{Re}(\beta) > 0$, (Gradshteyn and Ryzhik (1980), p. 318.).

Thus

$$\int_{-\infty}^{\infty}(A-2i\nu_0 B)^{-n/2}e^{-iu\nu_0}d\nu_0 = (2B)^{-n/2}\int_{-\infty}^{\infty}\left(\frac{A}{2B}-i\nu_0\right)^{-n/2}e^{-iu\nu_0}d\nu_0$$

$$= 2\pi u^{n/2-1}(2B)^{-n/2}\exp(-uA/2B)/\Gamma(n/2)$$

(since $\operatorname{Re}(A/B) > 0$, or else the density (2.8) would be degenerate, which would contradict the assumptions made in the previous section.)

Substituting (2.8) and the above into (2.5) and changing the order of integration, we now obtain:

$$p(\underset{\sim}{y}|Y(0)=0,\dot{Y}(0)\geq 0)_{h.w.} =$$

$$= (2\pi)^{-k}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{-\underset{\sim}{\nu}'\underset{\sim}{y}}\left\{\frac{e^{u/2}B^{-n/2}}{2\sqrt{2\pi\lambda u}}\int_0^{\infty}|z|\int_{-\infty}^{\infty}\exp[-\frac{uA}{2B}-i\xi z]d\xi dz\right\}\prod_{j=1}^{k}d\nu_j \; .$$

Since the expression in braces does not depend on $\underset{\sim}{y}$, it equals the characteristic function $\psi(\underset{\sim}{\nu})$ of (2.4), and the proof is complete.

Remark. Similarly, if we now write Y_u^* to denote the process Y conditioned (in the horizontal window sense) on a downcrossing of $Y(t)$ of the level u at $t=0$, then it is easy to check that Y_u^* has a representation identical to that of Y_u , with the single difference that in the density (2.2) the expression $(\underset{\sim}{v}'\underset{\sim}{w})^+$ should be replaced by $(-\underset{\sim}{v}'\underset{\sim}{w})^+$. Furthermore, in the characteristic function (2.4) the range of the z-integration should be changed to the negative half-line.

The density (2.2) turns out to be relatively awkward to work with, primarily because of the function $(\cdot)^+$ appearing there. Even the evaluation of the mean and covariance functions of the model process, is a difficult task, since it involves computation of the second and fourth order moments of $\underset{\sim}{V}$ and $\underset{\sim}{W}$. However, the joint characteristic function is more amenable for evaluation of the moments of $Y_u(t)$, and leads to

Corollary 2.1 The model process $Y_u(t)(Y_u^*(t))$ has the mean function

$$\left.\begin{matrix} m(t) \\ m^*(t) \end{matrix}\right\} = n + (u-n)R^2(t) + \dot{R}^2(t)/\lambda \mp R(t)\dot{R}(t)\sqrt{2\pi u/\lambda} \tag{2.10}$$

and the covariance function

$$\begin{aligned} \left.\begin{matrix} \rho(s,t) \\ \rho^*(s,t) \end{matrix}\right\} &= 2(n-2u)R^2(t)R^2(s) + 2nR^2(t-s) + 4(u-n)R(t)R(s)R(t-s) \\ &+ (2/\lambda)(2u-\pi u-2)R(t)R(s)\dot{R}(t)\dot{R}(s) + (4/\lambda)\dot{R}(t)\dot{R}(s)R(t-s) \\ &- (2/\lambda^2)\dot{R}^2(t)\dot{R}^2(s) \pm \sqrt{2\pi u/\lambda}\{2R(t)\dot{R}(t)R^2(s)+2R(s)\dot{R}(s)R^2(t) \\ &+ (1/\lambda)R(t)\dot{R}(t)\dot{R}^2(s) + (1/\lambda)R(s)\dot{R}(s)\dot{R}^2(t)-2R(s)\dot{R}(t)R(t-s) \\ &- 2R(t)\dot{R}(s)R(t-s)\}\ . \end{aligned} \tag{2.11}$$

Proof. The results follow via appropriate differentiation of (2.4) with k=1,2. Tedious calculations are omitted.

In order to compare the last result with the one obtained in the Gaussian case, we use the matched Gaussian process $Z(t)$ mentioned in the Introduction with mean and covariance function equal to those of $Y(t)$. For this process, the means of the model processes at upcrossings and downcrossings are given by

$$\left.\begin{matrix} m_z(t) \\ m_z^*(t) \end{matrix}\right\} = n + (u-n)R^2(t) \mp \sqrt{2n\pi/\lambda}\, R(t)\dot{R}(t) \text{ .}$$

To see how these differ from (2.10), Figure 1 gives the means of the model processes of a χ_4^2 process and a matched Gaussian process conditioned on a downcrossing of the level ½ at $t=0$. (Covariance function $R(t) = e^{-\frac{1}{2}t^2}$.) Note that the χ^2 process has a flatter, and higher, minimum than the matched Gaussian process. Both return reasonably quickly to the mean level of 4.

Figure 1 near here

We now turn to the study of the model processes Y_u and Y_u^* at high and low levels, respectively. In these cases the model processes take on particularly simple forms which are useful both in the modelling of χ^2 processes at extrema and for prediction purposes. We need to start with a technical result.

<u>Lemma 2.1</u> <u>Let</u> $\underset{\sim}{X} = (X_1,\ldots,X_m)'$, $\underset{\sim}{Y} = (Y_1,\ldots,Y_m)'$ <u>be independent random vectors;</u> $\underset{\sim}{X} \sim N(\underset{\sim}{0},a^2\underset{\sim}{I})$, $Y \sim N(\underset{\sim}{0},b^2\underset{\sim}{I})$. <u>Denote</u> $X = \underset{\sim}{X}'\underset{\sim}{X}$, $Y = \underset{\sim}{Y}'\underset{\sim}{Y}$, $Z = \underset{\sim}{X}'\underset{\sim}{Y}$. <u>Then the joint density of</u> X,Y,Z <u>is given by</u>

$$(2.12) \quad p(x,y,z) = C(xy - z^2)^{(m-3)/2}\exp\{-x/2a^2 - y/2b^2\} \ , \ x > 0, \ y > 0, \ xy-z^2 > 0;$$

$$C^{-1} = 2^m\sqrt{\pi}\,(ab)^m\Gamma(m/2)\Gamma((m-1)/2) \ .$$

<u>Proof</u>. The conditional ch.f. of Y,Z, given $\underset{\sim}{X} = \underset{\sim}{x}$, is

$$\psi_{Y,Z|\underset{\sim}{X}=\underset{\sim}{x}}(\nu_1,\nu_2) = E(\exp\{i\nu_1\underset{\sim}{Y}'\underset{\sim}{Y} + i\nu_2\underset{\sim}{x}'\underset{\sim}{Y}\})$$

$$= \prod_{k=1}^{m} E(\exp\{i\nu_1 Y_k^2 + i\nu_2 x_k Y_k\})$$

$$= \exp\{-i\nu_2^2\ \underset{\sim}{x}'\underset{\sim}{x}/4\nu_1\}\prod_{k=1}^{m} E(\exp\{i\nu_1(Y_k + \nu_2 x_k/2\nu_1)^2\}) \ .$$

For every $k=1,\ldots,m$ the r.v. $Y_k + \nu_2 x_k/2\nu_1$ has a $N(\nu_2 x_k/2\nu_1,b^2)$ distribution. Therefore the expectations in the last expression yield the ch.f. of a non-central χ_1^2 distribution:

$$E(\exp\{i\nu_1(Y_k+\nu_2 x_k/2\nu_1)^2\}) = (1-2i\nu_1 b^2)^{-1/2}\exp\{\frac{i\nu_2^2 x_k^2}{4\nu_1(1-2ib^2\nu_1)}\} \ .$$

Consequently

$$\psi_{Y,Z|\underset{\sim}{X}=\underset{\sim}{x}}(\nu_1,\nu_2) = (1-2i\nu_1 b^2)^{-m/2}\exp\{\frac{-\nu_2^2 b^2\ \underset{\sim}{x}'\underset{\sim}{x}}{2(1-2i\nu_1 b^2)}\} \ .$$

This being dependent on $\underset{\sim}{x}$ only via $\underset{\sim}{x}'\underset{\sim}{x} = x$, we can replace the conditioning event accordingly. Using the inversion theorem one obtains

$$p(y,z|x) = (2\pi)^{-2} \int_{-\infty}^{\infty} (1-2i\nu_1 b^2)^{-m/2} e^{-i\nu_1 y} \int_{-\infty}^{\infty} \exp -\{\frac{b^2 x \nu_2^2}{2(1-2i\nu_1 b^2)} - iz\nu_2\} d\nu_2 d\nu_1$$

$$= \frac{\exp\{-z^2/2b^2x\}}{b\sqrt{x}(2\pi)^{3/2}} \int_{-\infty}^{\infty} (1-2ib^2\nu_1)^{-(m-1)/2} \exp\{-i(y-z^2/x)\nu_1\} d\nu_1 \ .$$

Applying (2.9) to the above yields

$$p(y,z|x) = \frac{(y-z^2/x)^{(m-3)/2} \exp\{-y/2b^2\}}{\sqrt{\pi x}\, 2^{m/2} b^m \Gamma((m-1)/2)} \ , \ y > z^2/x \ (x > 0).$$

Finally, (2.12) follows on observing that X/a^2 is a χ_m^2 variate.

We are now in a position to prove the following result, which hinges on the representation (2.1) for Y_u and the corresponding representation for Y_u^* .

Theorem 2.2 Define the normalised and limit model processes

$$\tilde{Y}_u(t) := Y_u(t/\sqrt{u}) - u \ ; \ \tilde{Y}_\infty(t) := -\lambda t^2 + 2\sqrt{\lambda}\, Zt$$

$$\tilde{Y}_u^*(t) := (1/u) Y_u^*(\sqrt{u}\, t) \ ; \ \tilde{Y}_0^*(t) := \lambda \Psi t^2 - 2\sqrt{\lambda}\, Zt + 1$$

where $Z = \underset{\sim}{V}'\underset{\sim}{W}$ and $\Psi = \underset{\sim}{w}'\underset{\sim}{w}$. Then Z and Ψ are, respectively Rayleigh and χ_{n+1}^2 variables with joint density

$$p(z,\psi) = \frac{z(\psi - z^2)^{(n-3)/2} e^{-\psi/2}}{2^{(n-1)/2} \Gamma(\frac{n-1}{2})} \ , \ z > 0, \ \psi > z^2 \ .$$

Furthermore

(i) $P\{\lim_{u\to\infty} \sup_{|t|\le t_0} |\tilde{Y}_u(t) - \tilde{Y}_\infty(t)| = 0\} = 1$

(ii) $P\{\lim_{u\downarrow 0} \sup_{|t|\le t_0} |\tilde{Y}^*_u(t) - \tilde{Y}^*_0(t)| = 0\} = 1$.

Proof. (i) By (1.3)

$$R(t/\sqrt{u}) = 1 - \frac{\lambda t^2}{2u}(1+o(1))$$

$$\dot{R}(t/\sqrt{u}) = -\frac{\lambda t}{\sqrt{u}}(1+o(1))$$

as $u \to \infty$, where the o-terms tend uniformly to 0 for t in any bounded interval. From Theorem 2.1 we have

$$\begin{aligned}\tilde{Y}_u(t) = &-\lambda t^2(1+o(1)) + 2\sqrt{\lambda} t \underset{\sim}{V}'\underset{\sim}{W}(1+o(1)) \\ &+ \sum_{i=1}^{n}\{\Delta_i^2(t/\sqrt{u}) + 2\sqrt{u}[1 - \frac{\lambda t^2}{2u}(1+o(1))]V_i\Delta_i(t/\sqrt{u}) \\ &+ 2\sqrt{\frac{\lambda}{u}}\, t(1+o(1))W_i\Delta_i(t/\sqrt{u})\} .\end{aligned}$$

By (1.4) and (2.3) the $\Delta_i(t)$ have continuously differentiable sample paths with $\Delta_i(0) = \dot{\Delta}_i(0) = 0$, and thus $\Delta_i(t/\sqrt{u}) = o(t/\sqrt{u})$ as $u \to \infty$, uniformly for bounded t. It remains therefore to find the distribution of $Z = \underset{\sim}{V}'\underset{\sim}{W}$. But since this is equivalent to the h.w. conditional distribution of $\dot{Y}(0)/2\sqrt{u\lambda}$, given an upcrossing of a level u at t=0, its density is given by

(2.13) $p(z) = \mu^{-1}4u\lambda z p(u, 2\sqrt{u\lambda}\, z)$, $z > 0$,

where $p(\cdot,\cdot)$ is the joint density of $Y(0)$ and $\dot{Y}(0)$. Given $\underset{\sim}{X}(0) = \underset{\sim}{x}$, $\dot{Y}(0) = 2\underset{\sim}{x}'\dot{\underset{\sim}{X}}(0)$ is a normal zero mean r.v. with variance $4\lambda\underset{\sim}{x}'\underset{\sim}{x}$, and since this depends on $\underset{\sim}{x}$ only via $\underset{\sim}{x}'\underset{\sim}{x} = u$, we may conclude that $\dot{Y}(0)$, given $Y(0) = u$, has (conditionally) a $N(0,4\lambda u)$ distribution. Therefore

$$p(u,\dot{y}) = p(\dot{y}|u)p(u) = \frac{u^{(n-3)/2}e^{-u/2}\exp\{-\dot{y}^2/8\lambda u\}}{\sqrt{\pi\lambda}\, 2^{(n+3)/2}\Gamma(n/2)} ,$$

and a direct substitution into (2.13) leads to

$$p(z) = z\, e^{-z^2/2} , \; z > 0 ,$$

the desired result.

(ii) In a similar way, via (1.3) and (1.4), as $u \to 0$

$$R(\sqrt{u}\, t) = 1 - \frac{1}{2}\lambda u t^2(1+o(1))$$

$$\dot{R}(\sqrt{u}\, t) = -\lambda\sqrt{u}\, t(1+o(1))$$

$$\Delta_i(\sqrt{u}\, t) = o(\sqrt{u}\, t)$$

uniformly for bounded t. From the remark following Theorem 2.1 we can then deduce that

$$\tilde{Y}_u^*(t) = 1 + \lambda t^2 \underset{\sim}{W}'\underset{\sim}{W} + 2\sqrt{\lambda}t\underset{\sim}{V}'\underset{\sim}{W} + o(\sqrt{u}) .$$

The joint distribution of $Z = \underset{\sim}{V}'\underset{\sim}{W}$ and $\psi = \underset{\sim}{W}'\underset{\sim}{W}$

is equivalent to the h.w. conditional joint distribution of $\dot{Y}(0)/2\sqrt{u\lambda}$ and $\dot{\underset{\sim}{X}}'(0)\dot{\underset{\sim}{X}}(0)/\lambda$, given a downcrossing of the level u at $t=0$. Their joint density is

$$(2.14) \qquad p(z,\psi) = \mu^{-1}4u\lambda|z|p(u,2\sqrt{u\lambda}\, z,\, \psi)\ , \qquad z < 0$$

where $p(\cdot,\cdot,\cdot)$ is the joint density of $Y(0)$, $\dot{Y}(0)$ and $\dot{\underset{\sim}{X}}'(0)\dot{\underset{\sim}{X}}(0)/\lambda$. By Lemma 2.1, this density is

$$p(y,\dot{y},\psi) = \frac{(\lambda y\psi - \dot{y}^2/4)^{\frac{n-3}{2}} \exp\{-y/2 - \psi/2\}}{2^{n+1}\sqrt{\pi}\ \lambda^{n/2-1}\Gamma(n/2)\Gamma((n-1)/2)}$$

and a substitution into (2.14) concludes the proof.

The above theorem can be summarized as follows. Near an upcrossing of a high level u the χ^2 process can be approximated by a random parabola

$$-\lambda ut^2 + 2\sqrt{\lambda u}\ Zt + u\ .$$

Near a downcrossing of a low level u the χ^2 process can be approximated by

$$\lambda\Psi t^2 - 2\sqrt{\lambda u}\ Zt + u$$

which, because of the random coefficient Ψ, is of quite a different form to the Gaussian case.

(It is interesting to note that the model process in the Gaussian case takes a similar asymptotic form, viz.

$$-\frac{1}{2}\lambda ut^2 + \sqrt{\lambda}Zt + u,$$

with Z once again Rayleigh.)

It is intuitively obvious that low and high level crossings should be followed by extrema. The model processes for extrema should therefore have a very similar form to those for crossings. We shall derive the limiting model processes of extrema with the aid of the following result.

Lemma 2.2 Let $\underset{\sim}{X} = (X(t_1),\ldots,X(t_k))'$ where $X(t)$ is a stationary, Gaussian, zero mean process with covariance function $R(t)$ $(R(0)=1)$ and finite second and fourth spectral moments (λ and ν respectively). Then the conditional distribution of $\underset{\sim}{X}$, given $X(0) = x$, $\dot{X}(0)=\dot{x}$, $\ddot{X}(0)=\ddot{x}$ is multivariate normal with expectation

$$\Upsilon = \left\{\frac{\lambda\alpha R_i+\ddot{R}_i}{\lambda(\alpha-1)}\, x - \frac{\dot{R}_i}{\lambda}\,\dot{x} + \frac{\lambda R_i + \ddot{R}_i}{\lambda^2(\alpha-1)}\,\ddot{x}\right\}_{i=1,\ldots,k}$$

and covariance matrix

$$\left\{\rho(t_i,t_j)\right\}_{i,j=1,\ldots,k} = \Big\{R_{ij} - \frac{1}{\lambda(\alpha-1)}\,[\lambda\alpha R_i R_j+R_i\ddot{R}_j+R_j\ddot{R}_i$$

$$+ (\alpha-1)\dot{R}_i\dot{R}_j + \frac{1}{\lambda}\,\ddot{r}_i\ddot{r}_j]\Big\}_{i,j=1,\ldots,k}\ .$$

where $\alpha = \nu/\lambda^2$.

Proof. The covariance matrix of $\underset{\sim}{X}, X(0), \dot{X}(0), \ddot{X}(0)$ is

$$\left[\begin{array}{cccc|ccc} 1 & R_{12} & \cdots & R_{1k} & R_1 & -\dot{R}_1 & \ddot{R}_1 \\ R_{21} & 1 & \cdots & R_{2k} & R_2 & -\dot{R}_2 & \ddot{R}_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{k1} & R_{k2} & \cdots & 1 & R_k & -\dot{R}_k & \ddot{R}_k \\ \hline R_1 & R_2 & \cdots & R_k & 1 & 0 & -\lambda \\ -\dot{R}_1 & -\dot{R}_2 & \cdots & -\dot{R}_k & 0 & \lambda & 0 \\ \ddot{R}_1 & \ddot{R}_2 & \cdots & \ddot{R}_k & -\lambda & 0 & \nu \end{array}\right] = \left[\begin{array}{c|c} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} \\ \hline \underset{\sim}{S}_{21} & \underset{\sim}{S}_{22} \end{array}\right], \text{ say.}$$

From the basic properties of the multinormal distribution it follows that the conditional distribution of $\underset{\sim}{X}$ is normal with mean

$$\underset{\sim}{\gamma} = S_{12}S_{22}^{-1}(x,\dot{x},\ddot{x})'$$

and covariance matrix

$$\{\rho(t_i,t_j)\}_{i,j} = S_{11} - S_{12}S_{22}^{-1}S_{21} .$$

Remark. Lindgren (1970) derived a similar result for $\dot{x} = 0$.

Applying the arguments of Lindgren (1984) it follows from the above lemma that the model process obtained by conditioning $Y(t)$ on having a local extremum of height u at $t=0$ can be written as

$$(2.15) \qquad \sum_{i=1}^{n} \{\sqrt{u}\, R(t)V_i - \dot{R}(t)W_i/\sqrt{\lambda} + [\lambda R(t) + \ddot{R}(t)]H_i/\lambda\sqrt{\alpha-1} + K_i(t)\}^2 ,$$

where $K_i(t)$ are independent, normal, zero mean processes, independent of $\underset{\sim}{V}$, $\underset{\sim}{W}$, $\underset{\sim}{H}$, and with the common covariance function $\rho(s,t)$, of Lemma 2.2. The distribution of $\underset{\sim}{V}$, $\underset{\sim}{W}$, $\underset{\sim}{H}$ is the h.w. conditional distribution of $\underset{\sim}{X}(0)/\sqrt{u}$, $\dot{\underset{\sim}{X}}(0)/\sqrt{\lambda}$ and $[\ddot{\underset{\sim}{X}}(0) + \lambda\underset{\sim}{X}(0)]/\lambda\sqrt{\alpha-1}$, given an extremum of height u at $t=0$. Note that with probability 1 $\underset{\sim}{V}'\underset{\sim}{V} = 1$ and $\underset{\sim}{V}'\underset{\sim}{W} = 0$. Also, $\sqrt{u(\alpha-1)}\, \underset{\sim}{V}'\underset{\sim}{H} + \underset{\sim}{W}'\underset{\sim}{W} - u$ is restricted to negative or positive values, respectively, according to whether (2.15) comes from conditioning on a maximum or minimum. It was shown in Aronowich and Adler (1985) that, given $Y(0) = u$ and $\dot{Y}(0) = 0$, the random variables $\dot{\underset{\sim}{X}}'(0)\dot{\underset{\sim}{X}}(0)/\lambda$ and $\underset{\sim}{X}'(0)\ddot{\underset{\sim}{X}}(0)/\lambda\sqrt{u(\alpha-1)}$ are conditionally independent having χ^2_{n-1} and $-\sqrt{\frac{u}{\alpha-1}} + N(0,1)$ distributions respectively.

Furthermore, since

$$\ddot{R}(t) = -\lambda + (\nu/2)t^2 + o(t^2) \ , \ t \to 0$$

and

$$K_i(t) = o(t), \quad t \to 0$$

with probability 1, we have the following

<u>Theorem 2.2</u> <u>Let</u> $Y_u(t)$ <u>and</u> $Y_u^*(t)$ <u>be the h.w. model processes given maxima and minima of height</u> u <u>at</u> $t=0$ <u>respectively</u>. <u>Write</u>

$$\tilde{Y}_u(t) = Y_u(t/\sqrt{u}) - u$$

$$\tilde{Y}_u^*(t) = (1/u)Y_u^*(\sqrt{u}\ t) \ .$$

<u>Then</u>

$$\text{(i)} \quad P\{\lim_{u\to\infty} \sup_{|t|\le t_0} |\tilde{Y}_u(t) + \lambda t^2| = 0\} = 1$$

$$\text{(ii)} \quad P\{\lim_{u\downarrow 0} \sup_{|t|<t_0} |\tilde{Y}_u^*(t) - \lambda\Psi t^2 - 1| = 0\} = 1 \ ,$$

<u>where</u> $\Psi = \underset{\sim}{W}'\underset{\sim}{W}$ <u>is a</u> χ^2_{n+1} <u>random variable</u>.

<u>Proof</u>. Using the preceding remarks, the proof is straightforward.

3. THE LIMITING DISTRIBUTION OF LENGTH AND HEIGHT OF AN EXCURSION

As a consequence of Theorem 2.1 we can obtain the limiting distribution of the length and height of an excursion above high and below low levels. Let τ_u and τ_u^* be the lengths of h.w. excursions by $Y(t)$ above and below a level u, respectively, i.e.

$$\tau_u = \inf\{t > 0: Y_u(t) = u\}$$
$$\tau_u^* = \inf\{t > 0: Y_u^*(t) = u\}.$$

Theorem 3.1

(i) $P\{\lim_{u\to\infty} \sqrt{u}\,\tau_u = 2Z/\sqrt{\lambda}\} = 1.$

(ii) $P\{\lim_{u\downarrow 0} \tau_u^*/\sqrt{u} = \dfrac{2Z}{\sqrt{\lambda}\,\Psi}\} = 1.$

Proof. From Theorem 2.1 we have that with probability 1

$$\lim_{u\to\infty} \sqrt{u}\,\tau_u = \lim_{u\to\infty} \inf\{t > 0: Y_u(t/\sqrt{u}) = u\} = 2Z/\sqrt{\lambda}$$

$$\lim_{u\downarrow 0} \tau_u^*/\sqrt{u} = \lim_{u\downarrow 0} \inf\{t > 0: Y_u^*(\sqrt{u}\,t) = u\} = \frac{2Z}{\sqrt{\lambda}\,\Psi}.$$

By using the joint density of Z and Ψ, it is easy to obtain the density of $\xi = \dfrac{2Z}{\sqrt{\lambda}\,\Psi}$ as

$$p(\xi) = \frac{2^{(n+3)/2}\Gamma(\frac{n+3}{2})}{\lambda^{(n+1)/2} n!\,\xi^{n+2}}\ {}_1F_1\left(\frac{n+3}{2}, n+1, -\frac{2}{\lambda\xi^2}\right), \quad \xi > 0$$

where ${}_1F_1$ is a degenerate hypergeometric function.

Now let Q_u and Q_u^* be the heights of h.w. excursions by $Y(t)$ above and below a level u, respectively. Thus

$$Q_u = \max\{Y_u(t) - u: 0 \le t \le \tau_u\}$$

$$Q_u^* = \min\{u - Y_u^*(t): 0 \le t \le \tau_u^*\}.$$

Theorem 3.2

(i) $P\{\lim_{u\to\infty} Q_u = Z^2\} = 1.$

(ii) $P\{\lim Q_u^*/u = Z^2/\Psi\} = 1.$

Proof. We have, with probability 1,

$$\lim_{u\to\infty} Q_u = \lim_{u\to\infty} \max \{Y_u(t/\sqrt{u}) - u: 0 \le t \le 2Z/\sqrt{\lambda}\} = Z^2 ,$$

$$\lim_{u\downarrow 0} Q_u^*/u = \lim_{u\downarrow 0} \min\{1 - Y_u^*(\sqrt{u}\, t)/u: 0 \le t \le \frac{2Z}{\sqrt{\lambda}\,\Psi}\} = Z^2/\Psi$$

Note that Z^2 is an exponential r.v. with mean 2, and, as a simple calculation shows , $\zeta = Z^2/\Psi$ has the density

$$p(\zeta) = \frac{n-1}{2}\,\zeta^{(n-3)/2} , \; 0 \le \zeta \le 1 .$$

4. ASYMPTOTIC POISSON CHARACTER OF THE PROCESS OF LOW LEVEL DOWNCROSSINGS

It is well known that the high level upcrossings and (by symmetry) the low level downcrossings of a Gaussian process often behave, after appropriate normalization, like the points of a Poission process. This occurs under two conditions. One, a local condition, requires that the sample paths of the process be sufficiently smooth. The second is a mixing condition, generally expressed by requiring that the covariance function tends to zero fast enough at infinity.

Sharpe (1978) extended the upcrossing result to χ^2 processes under the mixing condition $R(t) = O(t^{-\alpha})$ for some $0 < \alpha < 1$ as $t \to \infty$, and

Lindgren (1980) sharpened this by requiring only $R(t)\ell n\ t \to 0$ as $t \to \infty$. In this section we shall look at the corresponding result for downcrossings of low levels. (Unlike the Gaussian case, this does not follow trivially from the high level result.) As did Sharpe, we shall model our proof very closely on Cramér and Leadbetter's (1967) presentation of the Gaussian case. In the end we shall require the stronger mixing condition $R(t) = O(t^{-\alpha})$ rather than the weaker $R(t)\ \ell n t \to 0$. We note now that this is not due to the "older" style of proof, but is due to the fact that the left hand tail of a χ^2 distribution has a power decay to zero, unlike the exponential decay of the right hand tail. We also note that although Berman (1983,1984) has a general theory of rare events that would seem to incorporate the next result, it turns out that verifying that our conditions fit into his framework is as much work as proving everything virtually from scratch.

To avoid needless repetition of considerable detail, we shall assume throughout the following that the reader is equipped with copies of both Sharpe (1978) and Cramér and Leadbetter (1967).

<u>Theorem 4.1</u> <u>Let the common covariance function</u> $R(t)$ <u>of the component Gaussian processes</u> $X_i(t)$, $i=1,\dots,n$, <u>satisfy</u>

(4.1) $$R(t) = 1 - \frac{\lambda}{2!}t^2 + \frac{\nu}{4!}t^4 + o(t^4) \text{ as } t \to 0,$$

(4.2) $$R(t) = O(t^{-\alpha}) \text{ for some } 0 < \alpha < 1 \text{ as } t \to \infty.$$

<u>Let</u> $D_u(\tau)$ <u>denote the number of downcrossings by</u> $Y(t)$ <u>of the level</u> u <u>in the interval</u> $(0,\tau)$, <u>and let</u> $\mu = ED_u(1)$. <u>Then for every fixed</u> $k=0,1,\dots,$

(4.3) $$\lim_{u\to 0} P[D_u(\frac{\tau}{\mu}) = k] = \frac{\tau^k e^{-\tau}}{k!}.$$

<u>Proof</u>. The (Gaussian) proof in Cramér and Leadbetter (1967) is set out in the form of five lemmas, three of which carry over with only minor modifications in notation: Lemma 1 uses merely the fact that the limit of

$$(4.4) \qquad \mu = \sqrt{\frac{\lambda}{\pi}} \left(\frac{u}{2}\right)^{\frac{n-1}{2}} \frac{e^{-u/2}}{\Gamma(n/2)}$$

is zero as $u \to 0$; Lemma 3 follows directly from Lemma 2; while the proof of Lemma 5 is a direct adaptation of Sharpe's (1978) proof of his result (4.20).

We are left therefore with the task of proving Lemma 2 and pointing out the differences in the proof of Lemma 4.

Lemma 2 can be expressed as

$$(4.5) \qquad Q_u = E\{D_u(t_1)[D_u(t_1)-1]\}/ED_u(t_1) \xrightarrow[u\to 0]{} 0$$

for an interval of length

$$t_1 = \frac{\tau[\mu^{-1}]}{\mu[\tau/\mu^{\beta+1}]}$$

where

$$(4.6) \qquad 0 < (k+4)\beta < \alpha < 1.$$

(For notation see Cramér and Leadbetter (1967)).

<u>Proof of (4.5)</u>. Using (4.5) of Sharpe (1978) and (12.2.10) of Cramér and Leadbetter (1967) we write

$$(4.7) \qquad Q_u \leq \frac{1}{\mu t_1}\int_0^{t_1} ds_1 \int_0^{t_1} ds_2 E[\dot{Y}^-(s_1)\dot{Y}(s_2)|Y(s_1)=Y(s_2) = u]p(u,u) ,$$

where $p(\cdot,\cdot)$ is the joint density of $Y(s_1)$ and $Y(s_2)$. The density $p(u,u)$ can be obtained by a straightforward modification of the expression given in Miller (1980), p. 162 for the 2-variate Rayleigh density function

$$(4.8) \qquad p(u,u) = \frac{u^{\frac{n}{2}-1} e^{-\frac{u}{1-R^2}} I_{\frac{n-2}{2}}\left(\frac{u|R|}{1-R^2}\right)}{(1-R^2)|R|^{\frac{n-2}{2}} \Gamma(\frac{n}{2})2^{\frac{n}{2}+1}} ,$$

where R stands for $R(s_1-s_2)$ and I_ν is the modified Bessel function of the first kind and order ν.

Substituting (4.8) and the result (Sharpe, p. 379)

$$E[\dot{Y}^-(s_1)\dot{Y}^-(s_2)|Y(s_1)=Y(s_2)=u] \leq 2u[\lambda(1-R^2)-\dot{R}^2]/(1-R^2)$$

into (4.7), we find

$$(4.9) \qquad Q_u \leq \frac{K}{t_1}\int_0^{t_1}\int_0^{t_1} \sqrt{u}\, e^{-\frac{u}{1-R^2}} \frac{\lambda(1-R^2)-\dot{R}^2}{(1-R^2)^2|R|^{\frac{n-2}{2}}} I_{\frac{n-2}{2}}\left(\frac{u|R|}{1-R^2}\right) ds_1 ds_2 ,$$

with K an unspecified positive constant.

Making the change of variables $s = s_2-s_1$ and denoting the integrand in (4.9) by $g(s)$ we obtain

$$Q_u \leq K \int_0^{t_1} g(s)ds$$

by the virtue of g being an even function.

An asymptotic expansion of I_ν for large arguments (Abramowitz and Stegun (1964), p. 377) gives that for some $x > 1$ there exists a $C > 0$, such that

$$I_\nu(x) < Ce^x/\sqrt{x} \quad .$$

Taking $x = \frac{u|R|}{1-R^2}$, we have that as long as

$$(4.10) \qquad 1 > |R(s)| > -\frac{u}{2} + \sqrt{\frac{u^2}{4} + 1} \quad ,$$

then

$$(4.11) \qquad I_{\frac{n-2}{2}}\left(\frac{u|R|}{1-R^2}\right) < C\left(\frac{u|R|}{1-R^2}\right)^{-1/2} \exp\left\{\frac{u|R|}{1-R^2}\right\} \; .$$

Let

$$s_0 = \inf\{s \geq 0: |R(s)| \leq -\frac{u}{2} + \sqrt{\frac{u^2}{4} + 1}\} \quad .$$

From condition (4.1) we deduce that there is a positive C_1 such that

$$(4.12) \qquad 1-C_1 s^2 > |R(s)| \qquad \text{for } 0 < s < s_0 \; .$$

Write

$$s^* = \left[\frac{1}{C_1}\left(1+\frac{u}{2} - \sqrt{\frac{u^2}{4} + 1}\right)\right]^{1/2} \quad .$$

Clearly $s_0 \leq s^*$ and $s^* \sim \sqrt{\frac{u}{2C_1}}$ as $u \to 0$.

Since

$$1 - C_1 s^2/2 < (1 + C_1 s^2/2)^{-1} \; ,$$

it follows from (4.12) that

$$(4.13)\qquad 1 + |R| < 2 - C_1 s^2 < \frac{2}{1+C_1 s^2/2} \quad \text{for} \quad 0 < s < s_0 .$$

Using (4.1), it can be shown that

$$(4.14)\qquad \frac{\lambda(1-R^2)-\dot{R}^2}{(1-R^2)^{3/2}} = \frac{\nu-\lambda^2}{2\sqrt{\lambda}} s + o(s) \text{ as } s \to 0 .$$

Applying (4.11), (4.13) and (4.14) we have

$$\int_0^{s_0} g(s)ds < K\int_0^{s_0} s(\sqrt{1+\frac{u^2}{4}}-\frac{u}{2})^{-1} \exp - \{\frac{u}{1+|R|}\}ds$$

$$< K \frac{s^*}{\sqrt{u}} [\Phi(s^*\sqrt{C_1 u/2} - \frac{1}{2}](\sqrt{1+\frac{u^2}{4}} - \frac{u}{2})^{-1}$$

$$\sim K[\Phi(u/2) - \frac{1}{2}] \to 0 \quad \text{as } u \to 0 ,$$

where $\Phi(\cdot)$ is the standard normal probability integral.

Furthermore, since $\exp\{-\frac{u}{1-R^2}\}$ is bounded away from zero, and

$$\frac{\lambda(1-R^2) - \dot{R}^2}{(1-R^2)^2} \to \lambda \qquad \text{as } s \to \infty \quad ,$$

(this follows from (4.2)), we have for the remaining part of our integral:

$$\int_{s_0}^{t_1} g(s)ds \le K\sqrt{u} \int_{s_0}^{t_1} |R|^{\frac{2-n}{2}} I_{\frac{n-2}{2}} (\frac{u|R|}{1-R^2}) ds$$

$$= K u^{\frac{n-1}{2}} \int_{s_0}^{t_1} Z(s)^{\frac{2-n}{2}} I_{\frac{n-2}{2}} (Z(s))ds ,$$

where $Z(s) = \dfrac{u|R(s)|}{1-R^2(s)}$.

Since

$$z^{-\nu}I_\nu(z) \to 2^{-\nu}/\Gamma(\nu+1) \quad \text{as } z \to 0 ,$$

(Abrahmowitz and Stegun (1964, p. 375), the last inequality becomes

$$\int_{s_0}^{t_1} g(s)ds < K(t_1-s_0)u^{\frac{n-1}{2}}.$$

From $t_1 \sim \mu^{\beta-1}$ as $u \to 0$ we have for sufficiently small u that $t_1 < 2\mu^{\beta-1}$, which leads to

$$\int_{s_0}^{t_1} g(s)ds < Ku^{\frac{n-1}{2}}(2\mu^{\beta-1} - s_0)$$
$$Ku^{\frac{n-1}{2}\beta} \to 0 \quad \text{as } u \to 0.$$

This completes the proof of Lemma 2.

The only modification in the proof of Cramér and Leadbetter's (1967) Lemma 4 to be made is the order of the probability of the event g_r (pp. 265-266). This event in our case becomes

$$g_r = \{Y(\nu_r q) \leq u\}$$

of probability

$$P\{g_r\} = O(u^{n/2}) .$$

To avoid confusion we will denote Cramér and Leadbetter's n by n'. Consequently the final part of Lemma 4 becomes

$$P\{D_k\} - P\{E_k\} = O(n'^{k+1}u^{n/2}) = O(u^{\frac{n}{2}(1-\beta(k+1))+\beta(k+1)/2})$$

$$\to 0 \quad \text{as } u \to 0 \;,$$

where of course, in the definitions of the above events, upcrossings are replaced by downcrossings and $\{\xi(\nu q) > u\}$ by $\{Y(\nu q) < u\}$.

5. LIMITING DISTRIBUTION OF EXTREME VALUES

The limiting extreme value distribution of $\max\limits_{0\leq t\leq T} Y(t)$ as $T \to \infty$ was derived by Sharpe (1978) who showed that

$$P[\tfrac{1}{2}\max_{0\leq t\leq T} Y(t) - u_T \leq z] \to e^{-e^{-z}} \quad \text{as} \quad T \to \infty$$

where

$$u_T = \log[\frac{\sqrt{\lambda/\pi}}{\Gamma(n/2)} T(\log T)^{\frac{n-1}{2}}] \;.$$

Thus the asymptotic distribution of $\max\limits_{0\leq t\leq T} Y(t)$ is of Type 1 (c.f. Leadbetter, Lindgren and Rootzén (1983)), precisely as in the Gaussian case. One expects that since χ^2 processes are bounded below, with power-type densities, the asymptotic distribution of $\min\limits_{0\leq t\leq T} Y(t)$ should belong to a different domain of attraction. This is in fact the case, as the following result indicates.

<u>Theorem 5.1</u> <u>The minimum of a</u> χ^2 <u>process has an asymptotic distribution of Type 3</u>. <u>Specificaly, under the conditions of Theorem 4.1</u>

$$(5.1) \qquad P\{v_T \min_{0\leq t\leq T} Y(t) > z\} \to e^{-z^{\frac{n-1}{2}}} \;,\; z \geq 0 \;,\; \underline{\text{as}} \quad T \to \infty \quad ,$$

where

$$(5.2) \qquad v_T = \frac{1}{2}\left[\frac{\sqrt{\lambda/\pi}}{\Gamma(n/2)}\,T\right]^{\frac{2}{n-1}} \quad .$$

Proof. It follows from the asymptotic Poisson character of the down-crossings process that

$$P\{\min_{0\le t\le T} Y(t) > u\} \to e^{-\tau} \quad \text{as } u \to 0 \quad ,$$

where u and T are related via

$$(5.3) \qquad \tau = \mu T \sim \frac{\sqrt{\lambda/\pi}}{\Gamma(n/2)} \left(\frac{u}{2}\right)^{\frac{n-1}{2}} T$$

for large T and small u.

Putting $\tau = z^{\frac{n-1}{2}}$ in (4.3) we obtain

$$u \sim 2\left[\frac{\Gamma(n/2)}{T\sqrt{\lambda/\pi}}\right]^{\frac{2}{n-1}} z \ ,$$

and (5.1) - (5.2) readily follow.

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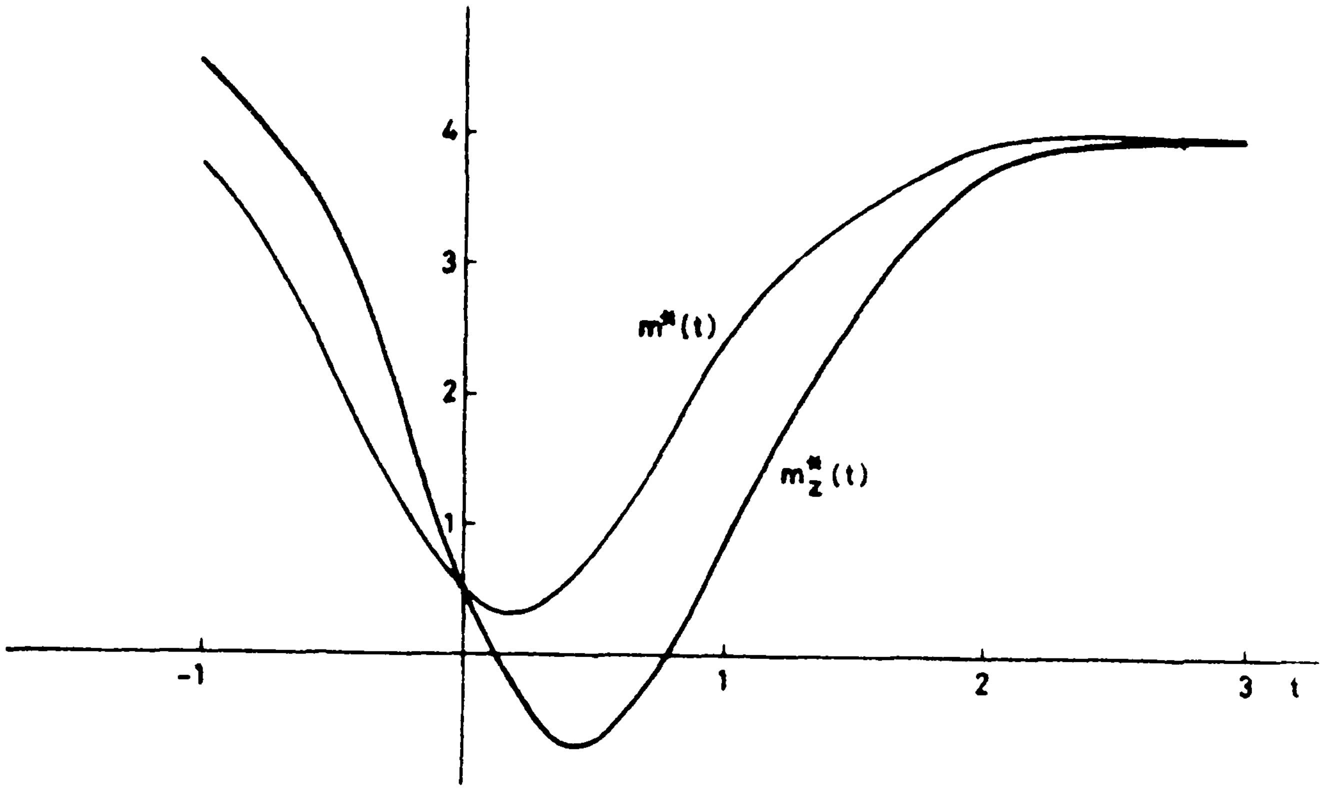

Figure 1: Mean functions for χ_4^2 and matched Gaussian processes in the neighbourhood of a downcrossing of ½ at t=0.

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